Asymmetric Dependence Implications for Extreme Risk Management

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Abstract

The fact that asset returns are more dependent in bear markets than in bull markets, called asymmetric correlation or dependence is well documented in financial literature. We analyze in this paper the implications of this phenomenon on the management of extreme risks. We show that in the presence of asymmetric dependence, a portfolio model based on a multivariate symmetric GARCH with Gaussian or Student $t$ innovations will lead to an underestimation of the portfolio value at risk (VaR) or expected shortfall. The latter will increase when negative returns become more dependent and positive returns less dependent, while the marginal distributions are left unchanged. In fact, we show that the strong dependence for low returns increases the downside risk and this additional risk cannot be captured by the Gaussian distribution. By introducing lower tail dependence, the Student-$t$ distribution corrects this shortcoming of the Gaussian distribution. However, the symmetric property of the Student-$t$ means also the same dependence in the upper tail and this will reduce the downside risk. The risk model that takes into account asymmetric dependence should allow lower tail dependence and upper tail independence as put forward by Longin and Solnik (2001). The Gumbel copula captures this asymmetry and shows superiority compared to Gaussian and student-$t$ while combined with DCC in terms of accuracy of extreme risk measures.

Keywords: Value at Risk, Expected Shortfall, Asymmetric Dependence, GARCH, DCC, Copula.

JEL classification: C52, C53, G10
1 Introduction

The value-at-risk (VaR) which defines the maximum loss on an investment over a specified time horizon at a given confidence level is become the key measure for financial risk used by many banks and financial institutions. Despite its widespread adoption by the financial community, the VaR concept faces some criticisms\textsuperscript{1}. To respond to these criticisms a complementary measure was proposed. Called expected shortfall (ES) it is the average loss when the VaR is violated. Both risk measures are very useful to manage extreme risk. However, the issue of their accurate estimation remains a challenge. In fact the estimates of VaR and ES depend on the distribution model. For the univariate models, Aas and Hobæk Haff (2006) find that GH skewed Student-\textit{t} distributions have a better data fit for the univariate skewed financial returns.\textsuperscript{2} In the multivariate parametric framework, the dynamic conditional correlation (DCC) of Engle (2002) is commonly used to model the dynamics of dependence. In this setup, the Gaussian or Student-\textit{t} distribution assumption is usually made. Although these models are easy to implement and can give satisfactory results in many situations, they can be seriously misleading in the case where data exhibit strong asymmetric dependence.

Actually, one of the most important empirical facts observed in multivariate financial returns, is a much stronger correlation between equity returns in bear markets than in normal or boom phases. This phenomenon known as asymmetric correlation or more generally asymmetric dependence has been analyzed by Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002), Garcia and Tsafack (2007), and references therein.

This paper examines the implications of ignoring this fact when implementing a model to estimate extreme risks such as VaR and ES. We show that by using multivariate Gaussian laws to characterize conditional distributions when a strong asymmetric dependence is present, one will underestimate VaR and ES. The strong dependence for low returns increases the downside risk and this additional risk cannot be captured by the Gaussian distribution. By introducing a lower tail dependence, the Student-\textit{t} distribution corrects this shortcoming of the Gaussian distribution. However, the symmetric property of the Student-\textit{t} implies the same dependence in both the upper and lower tails, and this will reduce the downside risk. The risk model that takes into account asymmetric dependence

\textsuperscript{1} Artzner et al. (1999) set four criteria that any coherent measure must satisfy and show that the VaR violates one of them. Basak and Shapiro (2001) also find that investors who just care about the VaR will take positions such that when the VaR is violated the loss can be very extreme.

\textsuperscript{2} The $\alpha$-stable distributions, which are similar to skewed-\textit{t} distributions but provide more flexibility in controlling the tails should also be good candidates to modeling univariate skewed data (see Garcia Renault and Veredas, 2004).
should allow lower tail dependence and upper tail independence as put forward by Longin and Solnik (2001).

Following Lee and Long (2002), we introduce asymmetric dependence into the DCC framework. The idea is that for Gaussian distributions, the zero correlation means independence, which is not the case for asymmetric dependence. To keep the DCC setup, we perform a transformation of the innovation vector to obtain uncorrelated variables and use a Student-\(t\) copula to capture tail dependence or, more importantly, a Gumbel copula to capture the asymmetry.

The empirical investigations using US and Canada equity and bond indices show that at the 5% level all three dependence specifications (Gaussian, Student-\(t\), and Gumbel) provide a good estimation of VaR. However at this same level, Gaussian and Student-\(t\) seriously underestimate the ES. For a more prudential level (1% and 0.5%), symmetric specifications tend to underestimate VaR more than they do for ES. The asymmetric dependence specification namely the Gumbel copula works well at all levels.

In the next section of this paper, we deal with the estimation of extreme risk in the multivariate framework, and provide theoretical arguments to explain why symmetric dependence tend to underestimate VaR and ES. Section 3 presents backtesting and underestimation test procedures, while the empirical analysis is carried out in section 4. Section 5 concludes.

2 Portfolio Risk and Dependence Structure Modeling

When measuring the risk of a portfolio, a nice approach should be to express it as an analytical function of individual risk levels for different components of this portfolio and their dependence parameters. It is the case for a risk measure as the variance of a portfolio which is completely defined by the variances of different components and the correlation between them. For extreme risk measures like VaR and expected shortfall, it is not always possible to write the risk level of a portfolio as a function of the risk levels of its different components. Although these extreme risk measures are simple to define, their accurate estimation can be very challenging. In fact, the VaR of a portfolio depends on two main components. The distributions of different single assets and the dependence structure between all individual assets. Since the VaR is a quantile, it takes into account the distribution shape contrary to the variance which is completely defined by just the second moment.

Formally, the value at risk \(VaR_p(X)\) of a variable \(X\) for a specified time horizon and a given level \(p\) is defined by: \(\Pr[X \leq -VaR_p(X)] = p\), and the related Expected Shortfall (ES) which is the average loss beyond the VaR is \(ES_p(X) = -E[X | X \leq -VaR_p(X)] = \)
\(-p^{-1}E[X.I (X \leq -VaR_p(X))]\).

When dealing with multivariate modeling, Gaussian or conditional Gaussian multivariate distributions are usually used. In this situation, the correlation coefficients completely characterize the dependence among the variables and the estimation of both VaR and ES is straightforward.

### 2.1 Gaussian Case

The most common and easy approach to modeling multivariate asset returns is to assume normality. In this case, the value at risk is completely defined by the first two moments. If \(X_t\) is a normal variable, the analytical expression of the value at risk is given by:

\[
VaR_p(X_t) = -\left(\mu_{X_t} + \Phi^{-1}(p)\sigma_{X_t}\right)
\]

with

\[
\begin{aligned}
\mu_{X_t} &= \left[w_t \mu_{1t} + (1 - w_t) \mu_{2t}\right] \\
\sigma_{X_t} &= \left[w_t^2 \sigma_{1t}^2 + (1 - w_t)^2 \sigma_{2t}^2 + 2w_t(1 - w_t)\rho_t \sigma_{1t} \sigma_{2t}\right]^{1/2} ,
\end{aligned}
\]

where \(\mu_{it}\) is the mean and \(\sigma_{it}\) the standard deviation of \(X_{it}\) and \(\rho_t\) the correlation coefficient between \(X_{1t}\) and \(X_{2t}\). So in the case of normal distributions, the VaR of a portfolio is expressed in closed form as a function of the parameters of the different single asset returns and the correlation coefficient between them. Moreover, both VaR and ES are related by the expression

\[
ES_p(X_t) + \mu_{X_t} = \Phi\left[\Phi^{-1}(p)\right] \left(VaR_p(X_t) + \mu_{X_t}\right).
\]

So, the expected shortfall is

\[
ES_p(X_t) = -\mu_{X_t} + \sigma_{X_t} \frac{\Phi\left[\Phi^{-1}(p)\right]}{p}
\]

where \(\phi\) denotes the density of a standard normal distribution.

### 2.2 Multivariate GARCH

Volatility clustering is an important stylized fact that should be taken into account when dealing with conditional distributions of asset returns. For univariate distributions, GARCH
models are commonly used to forecast the conditional volatility. The straightforward generalization of the univariate GARCH model brings some problems in the estimation process. Engle (2002) introduces a new class of multivariate GARCH models which is the generalization of the Bollerslev (1990) model. In fact, to allow tractability in multivariate GARCH, Bollerslev (1990) assumes a constant conditional correlation (CCC). The dynamic conditional correlation (DCC) of Engle (2002) extends this model by allowing time variation for correlation coefficients. As Engle (2002), we use a two-step estimation procedure for all models. In the first step, we estimate parameters of marginal distributions, and use them in a second step to estimate the parameters of the dependence structure.

2.2.1 Univariate Distribution Model

Lee and Long (2006) find that for multivariate models, the choice of copula functions is more important than the choice of the volatility models. Similarly, we will focus on the effect of the dependence structure on downside risk estimation. Therefore it is necessary to use for all different multivariate models the same marginal specification. For all single asset $x_{it}$, we use the simple GARCH(1,1) model.

\[
\begin{align*}
\left\{ & \begin{array}{l}
  x_{it} = \mu_{it} + \sigma_{it} \varepsilon_{it}, i = 1, 2 \\
  \sigma_{it+1}^2 = \omega_i + \beta_i \sigma_{it}^2 + \alpha_i (x_{it} - \mu_{it})^2
\end{array} \right.
\]

2.2.2 Dynamic Conditional Correlation (DCC)

Recently proposed by Engle (2002) to capture the time dynamics in the correlation, the DCC model has become a benchmark model for multivariate specifications. One of the attractive points of this model is its flexibility in terms of specification of the marginal distributions separately from the dependence structure. In our context of bivariate models, the GARCH(1,1)-type specification of the conditional correlation coefficient $\rho_t = \text{corr}(x_{1t}, x_{2t})$ is the following.

\[
\rho_t = \frac{q_{12,t}}{\sqrt{q_{11,t}q_{22,t}}}
\]

with the auxiliary variable $q_{ij,t}$ defined by the dynamic

\[
q_{ij,t+1} = \bar{p}_{ij} + a (\varepsilon_{it} \varepsilon_{jt} - \bar{p}_{ij}) + b (q_{ij,t} - \bar{p}_{ij}).
\]

The conditional value at risk $VaR^p_t(X_t)$ is defined by $\Pr[X_t \leq -VaR^p_t(X_t)|F_{t-1}] = p$, where $F_{t-1}$ is the information set available. For a Gaussian or a conditional Gaussian distribution, the importance of dependence is completely driven by the correlation coefficient.
However, for a more complex dependence structure, even if all components are individually conditional Gaussian (but not jointly), the portfolio VaR becomes more complex and depends on the shape of the dependence structure. The portfolio return distribution can be very complex. In fact, the distribution of a linear combination involves a convolution for which it is difficult to get an analytical expression. In such a context, it is common to use Monte Carlo simulations. Ultimately, the accuracy of estimation will depend on how well the dependence model fits the data.

By writing
\[
\begin{bmatrix}
x_{1t} \\
x_{2t}
\end{bmatrix} = \begin{bmatrix} \mu_{1t} \\ \mu_{2t} \end{bmatrix} + \begin{bmatrix} 1 \\ \rho_t \sqrt{1 - \rho_t^2} \end{bmatrix} \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix},
\]
we have \( \text{corr}(\eta_{1t}, \eta_{2t}) = 0 \). Then, if \((x_{1t}, x_{2t})\) is a bivariate normal vector, the zero correlation means that \( \eta_{1t} \) and \( \eta_{2t} \) are independent. However, if the joint normality is not valid as it is the case in many practical situations, the zero correlation does not necessarily mean independence. This is the case of asymmetric dependence.

### 2.3 Asymmetric Dependence Distributions and Extreme Risk Measures

Lower returns are more dependent than upper returns in financial markets, especially in international asset markets. Longin and Solnik (2001) investigate the structure of correlation between various equity markets in extreme situations and find that equity markets exhibit a much higher correlation in extreme bear periods and zero correlation for asymptotic upper returns. Garcia and Tsafack (2007) use tail dependence functions to extend this analysis in terms of nonlinear dependence and find similar results.

#### 2.3.1 Beyond Symmetric Dependence: Copula

Any bivariate distribution is defined by its marginal univariate distributions and its dependence structure between both variables. To completely characterize the dependence structure, we use copulas which are functions that build multivariate distribution functions from their unidimensional marginal distributions. Let \( X \equiv (X_1, X_2) \) be a vector of two variables. We denote by \( F \) the joint distribution function and by \( F_1 \) and \( F_2 \) the respective marginal distributions of \( X_1 \) and \( X_2 \). The Sklar theorem\(^3\) states that there exists a function \( C \) called copula which joins \( F \) to \( F_1 \) and \( F_2 \) as follows.

\[
F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \quad (2.1)
\]

\(^3\)See Nelsen (1999) for a general presentation. Note that if \( F_i \) is continuous for any \( i = 1, \ldots, n \) then the copula \( C \) is unique.
Equivalently the copula function $C$ is directly defined as follows.

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)); \quad u_1, u_2 \in [0,1]$$  \hspace{1cm} (2.2)

The nice thing about copulas is that we can model some stylized facts like asymmetric dependence without changing the DCC specification. By continuing to keep the zero correlation between $\eta_{1t}$ and $\eta_{2t}$, we can use the Gumbel copula to model their dependence structure.

### 2.3.2 Underestimation of VaR and ES

Practitioners usually use the Gaussian (or symmetric) distributions in the computation of the extreme risks in their portfolios. In fact, if data exhibit asymmetric dependence, the use of the Gaussian distribution will be misleading. The issue of interest is to know on which side will the dependence be stronger. In other words, if the extreme risk will be overestimated or underestimated. In this section we will develop some theoretical arguments that give us intuition about why the Gaussian dependence structure underestimates these risks.

By using copulas, any bivariate cumulative distributional function can be represented with three elements as $F \equiv (F_1, F_2, C)$ such that $F(X_1, X_2) = C(F_1(X_1), F_2(X_2))$, with $F_i$ be the cumulative distribution function of $X_i$, and $C$ the copula function of $(X_1, X_2)$. We want to analyze the effect of the third element which characterizes the dependence structure on the extreme risk measures. It is therefore relevant to keep the first and second elements which characterize the marginal distributions unchanged.

**Definition 1.** (Stochastic ordering,: $\prec_{st}$ Joe, 1997). $F' \prec_{st} F$ if $\int gdF' \leq \int gdF$ for all increasing functions $g$ for which the expectations exist.

The concept of stochastic ordering is equivalent to stochastic dominance in the case of a univariate distribution.\(^4\) The result below can be seen as an extension of the mean preserving spread to the multivariate case.\(^5\)

**Proposition 1.** Let $(X_1, X_2) \sim F \equiv (F_1, F_2, C)$ and $(X_1', X_2') \sim F' \equiv (F_1, F_2, C')$.

Denote $X = wX_1 + (1 - w)X_2$ and $X' = wX_1' + (1 - w)X_2'$ $w \in [0,1]$.

If $C' \prec_{st} C$ then

\(^4\)For univariate distributions $F$ and $F'$, by taking $g(v) = -I(v \leq x)$ which is an increasing function, we have $\int gdF' \leq \int gdF$ implies $F'(x) \geq F(x)$. So $F' \prec_{st} F$ is equivalent to $F' \geq F$.

\(^5\)Here we are focusing on bivariate distributions for the sake of presentation. The proof of this result in the appendix is valid for any dimension.
i) \( \text{VaR}_p (X') \geq \text{VaR}_p (X) \)

ii) \( \text{ES}_p (X') \geq \text{ES}_p (X) \)

Proof see Appendix

This result allows us to understand why the extreme risk measure in a left dependence asymmetric distribution should be larger than the one measured in a symmetric distribution with low dependence in the lower tail. The result below compares a normal copula with a rotated Gumbel copula.

Definition 2. (Tail dependence). For a copula \( C \), the lower tail dependence coefficient is

\[
\tau^L \equiv \lim_{u \to 0} \frac{C(u, u)}{u}
\]

while the upper tail dependence coefficient is

\[
\tau^U (\alpha) \equiv \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{u}
\]

The tail dependence coefficient measures the level of dependence between extreme events for both variables. For a Gaussian copula, regardless the level of correlation, both the upper and the lower tail dependences are zero. However, for a rotated Gumbel copula, the upper tail dependence is zero, while the lower tail dependence is strictly positive except in the case of independence. From this observation, we can compare both copulas in term of stochastic ordering.

Proposition 2. Let \( C_N \) be a normal copula and \( C_{rG} \) be a rotated Gumbel copula. If they are stochastically ordered,

then \( C_{rG} \preceq_{\text{st}} C_N \)

Proof see Appendix

It is not always possible to compare a Gaussian copula and a rotated Gumbel copula, however when it is the case, the rotated Gumbel copula is lower ordered than the Gaussian one. This result can not be straightforwardly extended to the comparison of a Student-\( t \) copula with a rotated Gumbel copula, since the tail dependence coefficient of a Student-\( t \) copula is not necessary zero. But the fact that a Student-\( t \) copula is symmetric suggests that a rotated Gumbel copula should put more probability in the left tail than a Student-\( t \) does. In fact, the level of tail dependence for a Student-\( t \) copula depends on the correlation
coefficient and the degree of freedom. For a zero correlation, the Student-\(t\) copula can have a tail dependence and this tail dependence decreases when the degree of freedom increases and tends to zero for an infinite degree of freedom, since it corresponds to the convergence of the Student-\(t\) distribution to the Gaussian distribution.

3 Testing and Comparison

It is important to perform a number of specification search to find the dependence functions which provide the best fit for a data set. Using a likelihood, Aikaike and Schwarz’s Bayesian information criteria we compare the dependence model goodness-of-fit. The accuracy of distribution models to estimate risk is assessed furthermore by backtesting using unconditional and conditional coverage. Finally we test the assumption of underestimation of VaR and ES in three dependence structure specifications (Gaussian, Student-\(t\) and Gumbel).

3.1 Backtesting

We present below the backtesting procedure for unconditional and conditional coverage. The first one tests the accuracy of violation probability assuming independence between successive violations, while conditional coverage also tests the independence.\(^6\)

3.1.1 Unconditional Coverage

Let \(I_t = I(X_t \leq -VaR_p(X_t))\) be the indicator of violation or not. The value 1 for \(I_t\) indicates that a violation occurred, while the value 0 means no violation. For a well specified risk model, the time series \(\{I_t\}_{t=1,...,T}\) should be time independent and identically distributed as Bernoulli with the probability \(p\) for 1. So we test the null hypothesis

\[
H_0 : I_t \sim i.i.d.\text{Bernoulli}(p)
\]

against

\[
H_{1uc} : I_t \sim i.i.d.\text{Bernoulli}(\pi), \pi \neq p
\]

The likelihood function is analytically derived and the likelihood ratio test can be easily performed. Denote

\[
\begin{align*}
T_1 &= \sum_{t=1}^{T} I_t \\
T_0 &= \sum_{t=1}^{T} (1 - I_t) = T - T_1
\end{align*}
\]

\(^6\)A complete presentation can be found in Christoffersen (2003).
the (log) likelihood ratio statistic is given by

\[ LR_{uc} = -2 \ln \left[ (1 - p)^{T_0} p^{T_1} / (1 - T_1/T)^{T_0} (T_1/T)^{T_1} \right] \sim \chi^2_1 \]

3.1.2 Conditional Coverage

The independence assumption can also be tested along with the probability of violation accuracy. Therefore the alternative hypothesis should include the time dependence between \( I_t \) and \( I_{t+h} \). We test

\[ H_0 : I_t \sim i.i.d. \text{ Bernoulli}(p) \]

against

\[ H_{1cc} : I_t \sim \text{Markov}(\Pi_1), \text{ with } \Pi_1 = \begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix} \]

By denoting \( T_{ij} \) the number of observations of \( I_t \) with an \( i \) value followed by a \( j \) value. Formally,

\[
\begin{align*}
T_{11} &= \sum_{t=1}^{T-1} I_t I_{t+1} \\
T_{10} &= \sum_{t=1}^{T-1} I_t (1 - I_{t+1}) \\
T_{01} &= \sum_{t=1}^{T-1} (1 - I_t) I_{t+1} \\
T_{00} &= \sum_{t=1}^{T-1} (1 - I_t) (1 - I_{t+1})
\end{align*}
\]

The likelihood ratio for this hypothesis is given by

\[ LR_{cc} = \begin{cases} 
-2 \ln \left( (1 - p)^{T_0} p^{T_1} / (1 - \hat{\pi}_{01})^{T_0} \hat{\pi}_{01} (1 - \hat{\pi}_{11})^{T_0} \hat{\pi}_{11} \right), & \text{if } \hat{\pi}_{11} \neq 0 \\
-2 \ln \left( (1 - p)^{T_0} p^{T_1} / (1 - \hat{\pi}_{01})^{T_0} \hat{\pi}_{01} \right), & \text{if } \hat{\pi}_{11} = 0
\end{cases} \]

where \( \hat{\pi}_{ij} = T_{ij} / (T_{i0} + T_{i1}) \) is the estimated probability of a \( j \) follows an \( i \). The asymptotic distribution of \( LR_{cc} \) is a \( \chi^2_2 \).

3.2 Testing the Extreme Risk Underestimation

For a risk manager, assessing the accuracy of the risk model is important and knowing if the model underestimates or not the risk is very important, since the underestimation can be very costly. We propose below simple ways to assess the underestimation of both value at risk and expected shortfall.
3.2.1 A Simple Test of VaR Underestimation

When the model is well specified the proportion of violations is exactly $p$. Formally by denoting $\pi = E[I_t]$, the expected proportion of violations with VaR computed under the true data generating process, we should have $\pi = p$. However, if the value at risk is underestimated, the expected number of violations will be larger than $p$, i.e. $\pi > p$. Then the testing hypothesis is

$$H_0 : \pi = p$$

against

$$H_1 : \pi > p$$

The estimator of $\pi$ is $\hat{\pi}_T = \frac{1}{T} \sum_{t=1}^{T} I_t$. Under the null, we have $I_t \sim i.i.d. \text{Bernoulli}(p)$. Therefore, it is a simple and well-known unilateral test for the mean of a variable.

3.2.2 A Simple Test of ES Underestimation

If we are using the correct risk model the expected shortfall should be exactly the mean of the returns in the portfolio tail with probability $p$. Therefore the regression of $(-X_t - ES_t) \times I(X_t \leq -VaR_p(X_t))$ on a constant term, will give zero coefficient. So, if we perform the regression

$$-X_t - ES_t = c + \varepsilon_t,$$

where $X_t \leq -VaR_p(X_t)$.

a correct model will produce zero for the estimate of $c$, while a model which underestimates the expected shortfall will produce $c > 0$. We then want to test the hypothesis

$$H_0 : c = 0$$

against

$$H_1 : c > 0$$

Once again, this is a simple unilateral test based on a Student-$t$ statistic. The estimate of the parameter $c$ is proportional to the difference between the sample mean and the model mean both computed for losses beyond the VaR.
4 Empirical Investigation

4.1 Data

We use equity and bond index data for US and Canada. The US equity returns are based on the SP 500 index, while the Canadian equity returns are computed with the Datastream index. The bond series are indices of five-year government bonds computed by Datastream. These bond indices are available daily and are chain linked allowing the addition and removal of bonds without affecting the value of the index. All returns are expressed in US dollar on a weekly basis from January 01, 1985 to December 21, 2004, which corresponds to a sample of 1044 observations.

To perform VaR forecasting, we split the full sample into two parts. The in-sample period starts from January 01, 1985 to December 23, 2003 and the out-of-sample period from December 23, 2003 to December 21, 2004. Descriptive statistics are provided in Tables 1&3.

4.2 Dependence Structure

Using the AIC and BIC criteria, we perform a comparison between two symmetric copulas (Gaussian and Student-t) and two asymmetric ones (Clayton and Gumbel). Table 4 shows that the Student-t copula is a better fit for our data than the Gaussian one, while the Gumbel copula is the best of all our compared dependence functions. From the sample correlation coefficients and the left tail dependence coefficients (TDC) computed from the Gumbel copula, one can notice that the dependence between equity indices is characterized by a higher level of correlation and also a stronger level of TDC than the dependence between bond indices.

4.3 Testing Results

To backtest our risk model, we perform both unconditional and conditional coverage tests for VaR estimation. Underestimation tests are done for both VaR and ES. All these tests are performed on an equally weighted US-Canada equity and bond portfolios.

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7The models are estimated in-sample and the out-of-sample data allow us to assess the forecasting ability of models. We perform backtests in all sample due to the limited size of out of sample data, however this is enough to take into account the out-of-sample effects.

8AIC and BIC refer to Akaike and Schwarz’s Bayesian information criteria respectively.

9Gumbel copula here refers to rotated (or survival) Gumbel copula. Since Gumbel copula has a left tail independence and right tail dependence, the rotated Gumbel copula exhibits a right tail independence and a left tail dependence that is the fact widely observed from data.
4.3.1 Equity Portfolio

Given the large correlation and strong asymmetric dependence between equity indices, their portfolio results are more relevant in terms of the asymmetric effect on risk estimation (see Table 5). The results show that at a 5% level all three copula models provide good VaR estimates. For this level of violation probability, there is no need to go beyond Gaussian DCC to estimate VaR even if it is important to notice that a good risk model for VaR at a specific level does not insure good estimation for ES at the same level. In fact, the ES depends on the entire shape of the distribution beyond the VaR at a given level. In other words, to see if a risk model will give a good estimation for ES or not based on VaR, one should extend the estimation to lower levels of violation probability. However, at a 1% violation level, the two symmetric models (Gaussian, and Student-t) fail the two tests at a 10% critical level, while the Gaussian model fails the tests at a 5% critical level. For lower violation level (0.5%) which represents in practice a very prudential risk management, the Gaussian does worse, while the Student-t improves significantly. The Gumbel copula model passes both tests at all probability levels.

4.3.2 Bond Portfolio

The dependence between bond indices is relatively low and therefore the DCC model seems to work regardless of the copula specification. Even if as shown in Table 6, the Gaussian copula risk model produces more violations at all probability levels, it passes all tests except only the unconditional coverage at a 10% critical level. The Student-t and Gumbel specifications perform statistically well.

4.3.3 Underestimation of VaR and ES

In the first panel of Tables 5 and 6, the number of violations $T_1$ is especially large for Gaussian dependence. Given this large frequency of violations observed with the sample estimation of coverage probability compared to the required level (5%, 1%, and 0.5%), we test the risk underestimation tendency of three models. Both VaR and ES measures are tested.

As shown in the p-value row of panel 3 in Table 5, the Gaussian model seriously underestimates the VaR for low levels (1% and 0.5%) for the equity portfolio, while the Student-t which underestimates the VaR at the 1% level gives a good estimation at 0.5%. Intuitively, this can be explained by the fact that the strong tail dependence of the Student-t corrects the effect of asymmetry for the lower tail distribution and reduces the magnitude
of underestimation. Surprising although explicable, is the fact that the ES is more likely underestimated for a large level (5%) (see the \( p \)-value row of panel 4, table 5 and 6). In fact all models provide a good estimate of VaR at the 5% level, but when the tails are not fat enough, the ES will be underestimated.

The Figure 1 clearly shows the ranking of the VaR level for the three dependence model specifications.\(^{10}\) The Gaussian model presents lower estimates of the VaR, while the Gumbel copula presents upper estimates. This is essentially due to the residual downside risk that the Gaussian copula cannot capture, while the Student-\( t \) partially incorporates it and the Gumbel copula takes it into account in a more effective way.

## 5 Conclusion

We provide arguments to explain the fact that symmetric dependence specifications tend to underestimate extreme risk in the presence of asymmetric dependence. In the DCC framework, we find that the Gaussian and Student-\( t \) specifications perform relatively well when the correlation or the tail dependence is low. However, in the presence of strong asymmetry like it is the case for equity indices, these symmetric specifications tend to underestimate VaR and ES. Therefore for the accuracy of risk measures, it is important in presence of asymmetric dependence to use an asymmetric model such as the Gumbel copula which allows lower tail dependence and upper tail independence.

We use bivariate models in this work, which is enough to show the effects of the dependence structure on the accuracy of risk measures. However, in a practical context to estimate portfolio risk, one deals with multivariate models of large dimension. It would be necessary to point out that the building of large dimension asymmetric copulas with more flexibility in the dependence structure among different couples remains a challenge in the statistical literature. The sensitivity to modeling asymmetry in the marginal distributions combine with asymmetric dependence would be another interesting issue.

\(^{10}\)We do not present the out-of-sample graphs for other violation levels (5% and .5%) because they have the same shape and lead to the same conclusions in terms of ranking than the 1% level. The ES ranking is exactly the same as the VaR ranking.
6 Appendix A

Proof of Proposition 1.

To prove this result, we need the below lemma.

**Lemma:** Let $F \equiv (F_1, F_2, C)$ and $(X_1, X_2) \sim F' \equiv (F_1, F_2, C')$.

$C' \preceq_{st} C$ is equivalent to $F' \preceq_{st} F$.

**Proof of lemma**

Let us assume that $C' \preceq_{st} C$ and let $g$ be an increasing function such that $\int gdF'$ and $\int gdF$ exist. we want to show that

$F' \preceq_{st} F$, i.e. $\int g(x_1, x_2) dF'(x_1, x_2) \leq \int g(x_1, x_2) dF(x_1, x_2)$

by defining $h(u, v) = g(F_1^{-1}(u), F_2^{-1}(v))$, since $F_i$ are increasing functions, $h$ is also an increasing function. Therefore $C' \preceq_{st} C$ implies

$\int h(u_1, u_2) dC'(u_1, u_2) \leq \int h(u_1, u_2) dC(u_1, u_2)$

i.e. \( \int g(F_1^{-1}(u_1), F_2^{-1}(u_2)) dC'(u_1, u_2) \leq \int g(F_1^{-1}(u_1), F_2^{-1}(u_2)) dC(u_1, u_2) \)

and therefore $F' \preceq_{st} F$.

Conversely by assuming $F' \preceq_{st} F$ and define $g(x, y) = h(F_1(x), F_2(y))$ we have the above lemma.

Q.E.D.

with the above lemma, we have by assumption that $F' \preceq_{st} F$, for $w \in [0, 1]$ by taking

i) $g(x_1, x_2) = -I(wx_1 + (1 - w)x_2 \leq -VaR_p(X))$, with $I(P) = 1$ if $P$ is true and 0 if not.

$g$ is an increasing function and we have

$\int g(x_1, x_2) dF(x_1, x_2) = -p \geq \int g(x_1, x_2) dF'(x_1, x_2)$

then

$-\int I(wx_1 + (1 - w)x_2 \leq -VaR_p(X')) dF'(x_1, x_2)$

$\geq -\int I(wx_1 + (1 - w)x_2 \leq -VaR_p(X)) dF'(x_1, x_2)$

i.e. $-VaR_p(X') \leq VaR_p(X)$ or $VaR_p(X') \geq VaR_p(X)$

ii) $g(x_1, x_2) = [wx_1 + (1 - w)x_2] I(wx_1 + (1 - w)x_2 \leq -VaR_p(X))$

$g$ is an increasing function since $VaR_p(X) \geq 0$, and we have

$ES_p(X) = -E[X \mid X \leq -VaR_p(X)]$

$= -p^{-1}E[X \mid X \leq -VaR_p(X)]$

$= -p^{-1} \int g(x_1, x_2) dF(x_1, x_2)$

$\leq -p^{-1} \int g(x_1, x_2) dF'(x_1, x_2)$

$= -p^{-1} \int [wx_1 + (1 - w)x_2] I(wx_1 + (1 - w)x_2 \leq -VaR_p(X)) dF'(x_1, x_2)$

$\leq -p^{-1} \int [wx_1 + (1 - w)x_2] I(wx_1 + (1 - w)x_2 \leq -VaR_p(X')) dF'(x_1, x_2)$

$= ES_p(X')$
Proof of Proposition 2.

Let $C_N$ be a normal copula and $C_{rG}$ be a rotated Gumbel copula that are "stochastically ordered", we want to show that $C_{rG} \prec_{st} C_N$. Let us suppose the reverse, i.e. $C_N \prec_{st} C_{rG}$.

Let $g(u_1, u_2) = -I(u_1 \leq u, u_2 \leq u)$ which is an increasing function. By assuming that $C_N \prec_{st} C_{rG}$ we have

\[ \int -I(u_1 \leq u, u_2 \leq u) \, dC_N(u_1, u_2) \leq \int -I(u_1 \leq u, u_2 \leq u) \, dC_{rG}(u_1, u_2) \]

so then

\[ \int I(u_1 \leq u, u_2 \leq u) \, dC_N(u_1, u_2) \geq \int I(u_1 \leq u, u_2 \leq u) \, dC_{rG}(u_1, u_2) \]

i.e.

$C_N(u, u) \geq C_{rG}(u, u)$.

However, since the tail dependence coefficient for normal copula $\tau^L_N = \lim_{u \to 0} \frac{C_N(u, u)}{u} = 0$ and for rotated Gumbel copula $\tau^L_{rG} = \lim_{u \to 0} \frac{C_{rG}(u, u)}{u} > 0$ then there exists $\eta \in (0, 1)$ such that for any $u \in (0, \eta)$

\[ \frac{C_{rG}(u, u) - C_N(u, u)}{u} > 0 \]

what is equivalent to $C_{rG}(u, u) > C_N(u, u)$. This contradicts the assumption $C_N \prec_{st} C_{rG}$, so we necessarily have $C_{rG} \prec_{st} C_N$.

Q.E.D.
References


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7 Tables and Graphs

Table 1: Summary statistics of weekly bond and equity index returns for US and Canada. All returns are expressed in US dollars on a weekly basis. The full sample is from January 01, 1985 to December 21, 2004; the in-sample period is from January 01, 1985 to December 23, 2003 and the out-of-sample period from December 23, 2003 to December 21, 2004. (δ Denotes annualized percent). Kurt represents the excess kurtosis of returns.

<table>
<thead>
<tr>
<th></th>
<th>Meanδ</th>
<th>Stdδ</th>
<th>Skewness</th>
<th>Kurt.</th>
<th>5 %VaR</th>
<th>1 %VaR</th>
<th>0.5 %VaR</th>
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<tr>
<td><strong>Panel A: full sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>US Equity</td>
<td>0.1367</td>
<td>0.1751</td>
<td>-1.5478</td>
<td>16.9169</td>
<td>0.0353</td>
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<td>0.0469</td>
<td>-0.0639</td>
<td>0.6628</td>
<td>0.0091</td>
<td>0.0153</td>
<td>0.0171</td>
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<td>0.1124</td>
<td>0.1672</td>
<td>-1.6671</td>
<td>13.5483</td>
<td>0.0372</td>
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<td>0.0815</td>
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<td>0.0173</td>
<td>0.0282</td>
<td>0.0327</td>
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<tr>
<td><strong>Panel B: in-sample</strong></td>
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<td><strong>Panel C: out-of-sample</strong></td>
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<td>US Equity</td>
<td>0.1029</td>
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<td>0.0423</td>
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<td>0.0261</td>
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Table 2: Estimates of GARCH (1, 1) parameters for all bond and equity returns. The figures between brackets represent standard deviations of the parameter estimates.

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<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
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<td></td>
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<td>(1.71e-05)</td>
<td>(3.27e-02)</td>
<td>(2.38e-02)</td>
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<td>8.32e-01</td>
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<tr>
<td></td>
<td>(3.41e-04)</td>
<td>(5.58e-06)</td>
<td>(5.42e-02)</td>
<td>(1.68e-02)</td>
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Table 3: Sample correlation coefficient (Corr) and left tail dependence coefficient (TDC) computed with the Gumbel copula.

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<td>TDC</td>
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<tr>
<td>Full Sample</td>
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<td>In Sample</td>
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<td>Out of Sample</td>
<td>0.6531</td>
<td>0.4864</td>
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Table 4: Akaike (AIC) and Schwarz’s Bayesian (BIC) information criteria for two symmetric copulas (Gaussian and Student-\textit{t} and two asymmetric copula (Clayton and Gumbel). Estimation is performed over the in-sample period.

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<td>Student-\textit{t}</td>
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<tr>
<td>US-CA Equities</td>
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<td>LogLikelihood</td>
<td>3.22e+02</td>
<td>3.29e+02</td>
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<td>AIC</td>
<td>-320.7354</td>
<td>-326.5728</td>
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<tr>
<td>BIC</td>
<td>-318.2604</td>
<td>-321.6229</td>
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<tr>
<td>US-CA Bonds</td>
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<tr>
<td>LogLikelihood</td>
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<tr>
<td>AIC</td>
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<td>BIC</td>
<td>-125.5840</td>
<td>-132.2591</td>
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Table 5: Test results over the full sample for the US and Canada equity 50/50 portfolio \((w = 0.5)\): the first panel presents the unconditional backtest. \(LR_{uc}\) is the likelihood ratio for unconditional coverage with a \(\chi^2_1\) distribution. In the second panel, \(LR_{cc}\) is the likelihood ratio for conditional coverage with a \(\chi^2_2\) distribution. \(\pi_T\) in the third panel is the sample estimation of violation probability which is used to test the underestimation. The last panel presents the test results for the ES. \(c\) is the difference between the sample mean of violations and the ES produced by the model. Three dependence models are tested. The Gaussian (Gauss.), the Student-\(t\) (\(t\)), and the Gumbel (Gumb.). Numbers in bold emphasize the statistical significance at the 5 or 10 percent level.

<table>
<thead>
<tr>
<th></th>
<th>5 %VaR</th>
<th></th>
<th>1 %VaR</th>
<th></th>
<th>0.5 %VaR</th>
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<tr>
<td></td>
<td>Gauss.</td>
<td>(t)</td>
<td>Gumb.</td>
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<td>(t)</td>
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<tr>
<td>(T_1)</td>
<td>56</td>
<td>52</td>
<td>50</td>
<td></td>
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<td>(LR_{uc})</td>
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<td>0.0945</td>
<td>5.7219</td>
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<td>1015</td>
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<td>(LR_{cc})</td>
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<tr>
<td>(\pi_T)</td>
<td>5.37%</td>
<td>4.99%</td>
<td>4.79%</td>
<td>1.82%</td>
<td>1.63%</td>
<td>1.34%</td>
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<tr>
<td>(\pi_T - p)</td>
<td>0.37%</td>
<td>-0.01%</td>
<td>-0.21%</td>
<td>0.82%</td>
<td>0.63%</td>
<td>0.34%</td>
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<tr>
<td>Std Err</td>
<td>0.0070</td>
<td>0.0067</td>
<td>0.0066</td>
<td>0.0041</td>
<td>0.0039</td>
<td>0.0036</td>
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<tr>
<td>(t) Stat.</td>
<td>0.5286</td>
<td>-0.0213</td>
<td>-0.3115</td>
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<td>1.6058</td>
<td>0.9601</td>
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<td>(p)-value</td>
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<td>0.5085</td>
<td>0.6223</td>
<td>0.0238</td>
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<tr>
<td>(c)</td>
<td>4.14e-4</td>
<td>4.14e-4</td>
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<tr>
<td>Std Err</td>
<td>2.25e-4</td>
<td>2.22e-4</td>
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<td>(t) Stat.</td>
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<td>0.1084</td>
<td>0.1284</td>
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Table 6: Test results over the full sample for the US and Canada bond 50/50 portfolio ($w = 0.5$): the first panel presents the unconditional backtest. $LR_{uc}$ is the likelihood ratio for unconditional coverage with a $\chi^2_1$ distribution. In the second panel, $LR_{cc}$ is the likelihood ratio for conditional coverage with a $\chi^2_2$ distribution. $\pi_T$ in the third panel is the sample estimation of violation probability which is used to test the underestimation. The last panel presents the test results for the ES. $c$ is the difference between the sample mean of violations and the ES produced by the model. Three dependence models are tested. The Gaussian (Gauss.), the Student-t ($t$), and the Gumbel (Gumb.). Numbers in bold emphasize the statistical significance at the 5 or 10 percent level.

<table>
<thead>
<tr>
<th></th>
<th>5 %VaR</th>
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<th>1 %VaR</th>
<th></th>
<th>0.5 %VaR</th>
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<tr>
<td></td>
<td>Gauss.</td>
<td>$t$</td>
<td>Gauss.</td>
<td>$t$</td>
<td>Gauss.</td>
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<tr>
<td>$T_{1}$</td>
<td>55</td>
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<td>$LR_{uc}$</td>
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<td>$LR_{cc}$</td>
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<tr>
<td>$p$-value</td>
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<tr>
<td>$\pi_T$</td>
<td>5.27%</td>
<td>4.51%</td>
<td>4.12%</td>
<td>1.25%</td>
<td>1.15%</td>
<td>1.05%</td>
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<tr>
<td>$\pi_T - p$</td>
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<td>-0.49%</td>
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<tr>
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<td>3.51e-5</td>
<td>3.29e-5</td>
<td>2.35e-5</td>
<td>2.24e-5</td>
<td>2.00e-5</td>
</tr>
<tr>
<td>$t$ Stat.</td>
<td>1.8133</td>
<td>2.0848</td>
<td>1.0225</td>
<td>1.8585</td>
<td>1.7061</td>
<td>1.1444</td>
</tr>
<tr>
<td>$p$-value</td>
<td>0.0350</td>
<td>0.0187</td>
<td>0.1534</td>
<td>0.0317</td>
<td>0.0441</td>
<td>0.1264</td>
</tr>
</tbody>
</table>

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Figure 1: Out-of-sample forecasts for the equity portfolio 1% VaR using the three dependence specification models.